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Asymptotic Properties of Coupled Nonlinear Langevin Equations in the Limit of Weak Noise. I: Cusp Bifurcation

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We show how a singular perturbation technique based on the introduction of properly scaled variables enables us to derive the asymptotic properties of coupled Langevin equations in the limit of weak noise. This technique can be applied when the macroscopic steady state is asymptotically or marginally stable. In the close vicinity of a cusp bifurcation point, a simple prescription for the adiabatic elimination of the fast variable is established. The critical variable exhibits amplified non-Gaussian fluctuations on a slow time scale. The properties of the fast variable depend on the nonlinearity of the system under consideration. Because of its coupling to the critical variable, it may exhibit amplified fluctuations of non-Gaussian nature.

KEY WORDS: Nonlinear stochastic differential equations; Fokker–Planck equations; fluctuations; bifurcations; adiabatic elimination.

1. INTRODUCTION

Stochastic processes have been applied to a large variety of problems, such as chemical systems, Brownian motion, parametric oscillators, population dynamics, hydrodynamics, quantum optics, and so on.[(1,2),4] For a Markovian process, the basic equations are the Master and the Fokker–Planck equations for discrete and continuous processes, respectively. Many studies have been devoted to the solution of these equations.[(3–8,29)] Among other

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techniques, singular perturbation has been successfully developed to extract relevant information from these equations in the case of one variable, both for homogeneous[9-12] and inhomogeneous systems.[13,14] Some results were also derived for a particular two variable model, the so-called Brusselator.[10,15-17]

In this paper we present a general discussion of the continuous stochastic processes for two-variable homogeneous systems, whether or not they obey detailed balance. In particular we verify, on the basis of a singular perturbation analysis, such procedures as the adiabatic elimination introduced by Haken.[2] Emphasis will be laid on the physical aspects of the techniques and results. Details of the analysis are reported elsewhere.[18] The most representative results will be shown in illustrative examples.

2. GENERAL CONSIDERATIONS

Consider the set of nonlinear Langevin equations:

$$\frac{d}{dt}x_i = f_i(x_1, \ldots, x_n) + \epsilon^{1/2}F_i, \qquad i = 1, 2, \ldots, n \tag{1}$$

where the $\{F_i\}$ are Gaussian white noises, defined by

$$\langle F_i(t)F_j(t')\rangle = Q_{ij}\delta(t-t') \tag{2}$$

The equivalent formulation in terms of the Fokker–Planck equation reads

$$\partial_t P(\{x_i\};t) = \sum_i \left(-\frac{\partial}{\partial x_i} f_i + \frac{\epsilon}{2}\sum_j Q_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\right) P(\{x_i\};t) \tag{3}$$

In this paper we study the asymptotic properties of the stochastic process $(x_1, \ldots, x_n) \equiv \{x_i\}$ defined by (1), in the limit of weak noise $\epsilon \to 0$. Apart from its mathematical interest, such a study is relevant for many physical problems. When the variables $\{x_i\}$ represent intensive thermodynamic variables, such as the chemical concentrations, temperature, and so on, the noise represents the effect of local thermal disturbances and hence it is proportional to the inverse-volume size, which is obviously small. In the case of Brownian motion, the noise is proportional to the (small) ratio of masses of Brownian particle and solvent molecules. In electric circuits, the noise comes from parasite effects which one tries to minimize. Although small, the noise may induce *macroscopic effects* when the stability of the system is threatened. This happens, for instance, in the vicinity of bifurcation points.

When one or several of the $\{x_i\}$ variables obey a closed subset of equations, they can be considered as external noise acting on the remaining variables.[28] It should be noted, however, that many phenomena related to external noise (such as noise-induced transitions[19]) cannot occur in the weak noise limit analyzed in the present paper.

Some general results concerning (1) are known. If one sets $\epsilon = 0$, the noise terms vanish and (1) reduces to the set of "macroscopic" equations:

$$\frac{d}{dt}\bar{x}_i = f_i(\bar{x}_1, \ldots, \bar{x}_n), \qquad i = 1, \ldots, n \tag{4}$$

Kurtz showed that, as far as an initial value problem is concerned, a small noise affects the process only perturbatively: stochastic trajectory $\{x_i(t)\}$ and macroscopic trajectory $\{\bar{x}_i(t)\}$ remain "close" to each other for some finite time:

$$|x_i(t) - \bar{x}_i(t)| \sim O(\epsilon^{1/2}), \qquad \forall t < T(\epsilon) \tag{5}$$

almost surely, if initially so.[20,21]

In a more physical context, this result states that a Gaussian probability density centered around the macroscopic state propagates in time. The above result holds for any nonlinearity of the system, but it is valid only for finite time: it leaves open the problem of the stationary state and the approach towards it.

To illustrate the difficulties that occur in the long time limit, let us consider the following example:

$$\frac{d}{dt}x = \lambda x - x^3 + \epsilon^{1/2}F \tag{6}$$

In the absence of the noise ($F = 0$), Eq. (6) has a bifurcation point at $\lambda = 0$ (see Fig. 1): for $\lambda < 0$ it possesses a unique stationary solution ($\bar{x} = 0$), while for $\lambda > 0$, two additional solutions ($\bar{x}_{\pm} = \pm\sqrt{\lambda}$) appear.

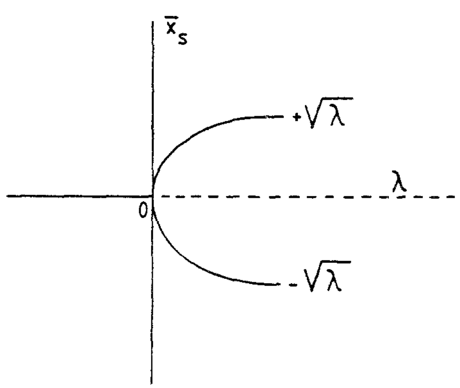

Fig. 1. Typical bifurcation diagram for the cusp transition.

Let us start by assuming F to be constant. Then one can write the series expansions

$$x(t) = \bar{x}(t) + \epsilon^{1/2}x_1(t) + \epsilon x_2(t) + \cdots \tag{7}$$

Clearly,

$$|x(t) - \bar{x}(t)| \sim O(\epsilon^{1/2}), \qquad \forall t < T, T < \infty \tag{8}$$

For $\lambda < 0$, the series (7) and the corresponding result (8) remain valid for $t \to \infty$, including the stationary state. For $\lambda = 0$, however, the stationary state displays another ϵ dependence:

$$|x_s - \bar{x}_s| \sim O(\epsilon^{1/6}), \qquad \bar{x}_s = 0 \tag{9}$$

The limits $\epsilon \to 0$ and $t \to \infty$ therefore do not commute. The same problem arises for $\lambda > 0$, i.e., in the region of coexisting macroscopic steady states.

The problem we raise here is the matching of different macroscopic regimes. Of course for the model (6), with constant F, the exact time-dependence solution is known. But the problem is much more involved when F is time dependent, *a fortiori* a stochastic function, and remains largely unsettled despite recent significant progress.[22–26]

The example (6) has one particular property which remains true for $\lambda = 0$, even in the limit $t \to \infty$:

$$\lim_{\epsilon \to 0} |x(t) - \bar{x}(t)| = 0, \qquad \begin{array}{l} \lambda \leqslant 0 \\ \forall t, \text{ including } t \to \infty \end{array} \tag{10}$$

This result is part of a more general statement which was formulated for systems involving one variable[14]: stochastic and macroscopic trajectory converge[5] in the limit $\epsilon \to 0$ for all times, including the stationary state, if the macroscopic steady state is unique and globally stable. This includes the case of marginal stability which in our example, Eq. (6), occurs for $\lambda = 0$.

In this paper we derive the asymptotic properties of the general process described by Eq. (1) in the limit $\epsilon \to 0$, at or near the stationary state and this, independent of whether or not the system obeys detailed balance. The transient regime and its corresponding matching problems will not be considered, although one situation where Kurtz results (5) can be extended to $t \to \infty$ will be discussed.

To illustrate our method, consider again the model (6) but with F a Gaussian white noise, defined by

$$\langle F(t)F(t') \rangle = Q\delta(t - t') \tag{11}$$

[5] The convergence was shown to be in probability, although a stronger law of convergence is expected.[27]

Since the problem involves one variable only, the stationary solution can easily be found:

$$P_s(x) \sim \exp\left[\frac{2\epsilon^{-1}}{Q} \left(\lambda \frac{x^2}{2} - \frac{x^4}{4} \right) \right] \tag{12}$$

In agreement with our general statement, the stochastic values converge (in probability) to the macroscopic stationary state $\bar{x}_s = 0$, when the latter is unique and globally stable ($\lambda \leqslant 0$):

$$\lim_{\epsilon \to 0} P_s(x) = \delta(x), \qquad \lambda \leqslant 0 \tag{13}$$

To evaluate the amplitude of fluctuations in the limit $\epsilon \to 0$, the perturbative expansion (7) suggests the introduction of a scaled variable:

$$u = \epsilon^{-1/2} x \tag{14}$$

This leads to

$$\lim_{\epsilon \to 0} P_s(u) = \left(\frac{-\lambda}{\pi Q} \right)^{1/2} \exp \frac{\lambda u^2}{Q} \tag{15}$$

at least when $\lambda < 0$. For $\lambda = 0$, $P_s(u)$ is no longer a normalizable probability density in the limit $\epsilon \to 0$ (since it vanishes for all u). The marginal stability of the macroscopic reference state is responsible for the amplification of the fluctuations and another ϵ dependence of the stationary solution is expected. Indeed, for

$$u = \epsilon^{-1/4} x \tag{16}$$

we find

$$\lim_{\epsilon \to 0} P_s(u) = \frac{2(2Q)^{-1/4}}{\Gamma(1/4)} \exp\left(-\frac{u^4}{2Q} \right), \qquad \lambda = 0 \tag{17}$$

When the explicit solution of the Langevin equation is not known, the introduction of the probability density for appropriately scaled variables can thus be helpful for studying the asymptotic properties of the process.(13,14) In general, if we set

$$u = \epsilon^{a-1} x; \qquad 0 \leqslant a < 1 \tag{18}$$

three typical situations can be distinguished:

$$\lim_{\epsilon \to 0} P_s(u) = \begin{cases} \text{does not exist}, & \text{if } a < a_c \\ f(u), & \text{if } a = a_c \\ \delta(u), & \text{if } a > a_c \end{cases} \tag{19}$$

where $f(u)$ represents a nontrivial (i.e., non-delta-function) probability density. The value $a = a_c$ defines the scaling which is relevant for the study of the fluctuations. Of course it may happen that such a value $a_c < 1$ does not exist. In our example, this is the case when $\lambda > 0$. In fact, in this case the macroscopic steady state $\bar{x}_s = 0$ loses its stability in favor of the two new solutions $\bar{x}_{\pm} = \pm\sqrt{\lambda}$ that arise. The probability function $P_s(x)$ is a two-humped density with a minimum at the unstable solution. For $\epsilon \to 0$, it reduces to the sum of two delta functions centered, respectively, at the stable macroscopic steady states.[10,11] This situation will not be discussed in this paper.

Let us now apply these ideas to two variable systems for which the stationary solution is not generally known. Let

$$\partial_t x = f(x, y) + \epsilon^{1/2} F_x \qquad \partial_t y = g(x, y) + \epsilon^{1/2} F_y \tag{20}$$

F_x and F_y are Gaussian white noises with correlation functions

$$\begin{aligned} \langle F_x(t) F_x(t') \rangle &= Q_{xx}\delta(t - t') \\ \langle F_x(t) F_y(t') \rangle &= \langle F_y(t) F_x(t') \rangle = Q_{xy}\delta(t - t') \\ \langle F_y(t) F_y(t') \rangle &= Q_{yy}\delta(t - t') \end{aligned} \tag{21}$$

We will suppose that f and g are analytic functions in the neighborhood of the macroscopic steady state $(\bar{x}_s, \bar{y}_s)$ which we take to be unique and globally stable (but not necessarily asymptotically stable). Then, the deviations $\delta x = x - \bar{x}$ and $\delta y = y - \bar{y}$ go to zero, in some sense, in the limit $\epsilon \to 0$ for all times including $t \to \infty$. Following the arguments developed in the case of one-variable problems, we introduce the scaled variables:

$$u = \epsilon^{a-1}\delta x \qquad v = \epsilon^{b-1}\delta y, \qquad a, b < 1 \tag{22}$$

where a and b should be chosen such that the corresponding probability density $P(u, v; t)$ remains normalizable in the limit $\epsilon \to 0$. Moreover, we require that its moments $\mu_{m,n}$,

$$\mu_{m,n} = \iint_{-\infty}^{\infty} du\, dv\, u^m v^n P(u, v; t), \qquad m, n = 0, 1, 2, \ldots \tag{23}$$

remain finite in the limit $\epsilon \to 0$, if they exist for $\epsilon \neq 0$. In order to avoid the trivial result $P(u, v; t) = \delta(u)\delta(v)$, we finally require that at least one of the moments $\mu_{m,n}$ be different from zero ($m + n \geqslant 1$). Using the above conditions, it can be proven that the scaling exponents a and b must obey the following inequality:

$$\tfrac{1}{2} \leqslant a, b < 1 \tag{24}$$

A general proof of this result is given in Ref. 18. We discuss its implications in detail later on.

The Fokker–Planck equation corresponding to the Langevin equations (20) reads

$$\partial_t P(x, y; t) = \left\{ -\frac{\partial}{\partial x} \left[\sum_{i,j=0}^{\infty} \bar{f}_{ij}(x - \bar{x})^i (y - \bar{y})^j \right] - \frac{\partial}{\partial y} \left[\sum_{k,l=0}^{\infty} \bar{g}_{kl}(x - \bar{x})^k (y - \bar{y})^l \right] + \frac{\epsilon}{2} \left[\frac{\partial^2}{\partial x^2} Q_{xx} + 2 \frac{\partial^2}{\partial x \partial y} Q_{xy} + \frac{\partial^2}{\partial y^2} Q_{yy} \right] \right\} P(x, y; t) \tag{25}$$

where $\bar{F}_{ij}$ stands for

$$\frac{1}{i!\, j!} \frac{\partial^{i+j}}{\partial x^i \partial y^j} F \bigg|_{x=\bar{x}, y=\bar{y}}$$

A Gaussian initial condition imposes $a = b = 1/2$. Equation (25) then reduces, to dominant order in ϵ, to a linear Fokker–Planck equation. In accordance with Kurtz's result, this initial Gaussian will propagate in time. The dynamics of the fluctuations is determined by the linearized operator

$$L = \begin{pmatrix} \bar{f}_{01} & \bar{f}_{10} \\ \bar{g}_{01} & \bar{g}_{10} \end{pmatrix} \tag{26}$$

The Gaussian law remains valid for $t \to \infty$, if the fluctuations of the scaled variables do not diverge in this limit. This depends on the eigenvalues of L, which determine the stability properties of the macroscopic stationary states. If the real parts of both eigenvalues are strictly negative, the fluctuations do not diverge and the distribution remains normalisable for $t \to \infty$. This leads us to the conclusion that the Gaussian law remains valid up to the stationary state when the macroscopic steady state is not only globally but also asymptotically stable. When the real part of an eigenvalue vanishes, the flucuations of at least one of the scaled variables diverges at the stationary state, indicating that the Gaussian scaling cannot be extended to $t \to \infty$. As is clear from the example (6) treated above, we expect a qualitative change in the properties of fluctuations.

It is known from the theory of nonlinear differential equations that the linear instability of the steady state indicates the possibility of bifurcation.[1] Three situations can be distinguished:

(i) One of the eigenvalues vanishes and the other remains negative. From the standpoint of bifurcation theory, the most interesting example of these situations is the case of a cusp bifurcation. It will be considered in detail in the next sections.

(ii) Two complex conjugated eigenvalues have a vanishing real part, but nonvanishing imaginary part. This is the case of a Hopf bifurcation. Its study is reported in a forthcoming paper.

(iii) The two eigenvalues vanish together. In this case no general conclusion can be drawn and one has to proceed to a nonlinear stability analysis. Such cases will not be considered in this paper.

3. MARGINAL STABILITY: REAL EIGENVALUES

We consider the case where the matrix L has two negative eigenvalues λ_1 and λ_2, one of them being possibly zero:

$$\lambda_1 \ll \lambda_2 \lesssim 0 \tag{27}$$

For simplicity we will assume that the deterministic variables are at their stationary state[6] $(\bar{x}_s, \bar{y}_s)$. Note that since the eigenvalues λ_1 and λ_2 are different, the matrix L can always be diagonalized. We can thus, without loss of generality, rewrite (25) in terms of the scaled variables (22) as

$$\begin{aligned}\partial_t P(u,v;t) = \Bigg\{ & -\frac{\partial}{\partial u}\left[\lambda_1 u + \sum_{i+j\geq 2} \bar{f}_{ij}\epsilon^{j(1-b)+(i-1)(1-a)} u^i v^j\right] \\ & -\frac{\partial}{\partial v}\left[\lambda' \epsilon^c v + \sum_{k+l\geq 2} \bar{g}_{kl}\epsilon^{k(1-a)+(l-1)(1-b)} u^k v^l\right] \\ & + \frac{\epsilon^{2a-1}}{2} Q_{xx}\frac{\partial^2}{\partial u^2} + \frac{\epsilon^{2b-1}}{2} Q_{yy}\frac{\partial^2}{\partial v^2} \\ & + \epsilon^{a+b-1} Q_{xy}\frac{\partial^2}{\partial u\,\partial v}\Bigg\} P(u,v;t)\end{aligned} \tag{28}$$

where, in order to explore the vicinity of the critical point $\lambda_2 = 0$, we have set $\lambda_2 = \epsilon^c \lambda'$ with fixed negative λ'. Of course for $c = 0$ the Gaussian scaling $a = b = 1/2$ prevails at the stationary state, as we discussed earlier. One then obtains from (28)

$$\langle \delta u^2 \rangle = -\frac{Q_{xx}}{2\lambda_1} \qquad \langle \delta v^2 \rangle = -\frac{Q_{yy}}{2\lambda_2} \qquad \langle \delta u\, \delta v \rangle = -\frac{Q_{xy}}{\lambda_1 + \lambda_2} \tag{29}$$

On the other hand, sufficiently close to the bifurcation point, i.e., for c sufficiently large, a change of behavior of the system is expected (the value

[6] The combined relaxation of the deterministic values and fluctuations leads to similar equations but with time-dependent coefficient.[29]

of c delimitating these two situations can be derived explicitly[18]). Indeed, for $\lambda_2 \to 0$ we observe the divergence of the (Gaussian) scaled variable v. This suggests that at the stationary state the fluctuations of the y variable are of lower order than $O(\epsilon^{1/2})$, and therefore we should choose a different value for b, larger than $1/2$. Although (29) predicts that the fluctuations of the x variable are not affected, one expects intuitively that the latter flucutations may by "contamination" also be amplified. We will return to this point in the next section.

It can be proven that values of a and b satisfying the conditions discussed in the last section [Eq. (24)] obey the following inequalities[18]:

$$a < b \tag{30}$$

In order to investigate the critical properties of the y variable, we derive the equation for $P(v, t)$ by integrating (28) over the u variable:

$$\partial_t P(v;t) = \left\{ -\frac{\partial}{\partial v}\left[\lambda' \epsilon^c v + \sum_{k+l \geq 2} \bar{g}_{kl} \epsilon^{k(1-a)+(l-1)(1-b)} \langle u^k \mid v \rangle v^l \right] + \frac{\epsilon^{2b-1}}{2} Q_{yy} \frac{\partial^2}{\partial v^2} \right\} P(v;t) \tag{31}$$

To obtain a closed equation for v, we have to calculate the conditional averages

$$\langle u^k \mid v \rangle = \int P(u \mid v;t) u^k \, du \tag{32}$$

where $P(u \mid v;t) = P(u,v;t)/P(v;t)$ is the conditional probability for u given a value of v. Now, from (24) it follows that the inequality $j(1-b) + (i-1)(1-a) \leq 0$, $i+j \geq 2$ can only hold for $i = 0$, whereas the inequality $k(1-a) + (l-1)(1-b) > 0$ for $k + l \geq 2$ always holds. Hence, by inspection of the orders of magnitude of ϵ in Eq. (28), we conclude that

$$\partial_t P(u,v;t) = \left\{ -\frac{\partial}{\partial u}\left[\lambda_1 u + \bar{f}_{0j} \epsilon^{j(1-b)-(1-a)} v^j \right] + \frac{\epsilon^{2a-1}}{2} Q_{xx} \frac{\partial^2}{\partial u^2} \right\} P(u \mid v;t)[1 + o(1)] \tag{33}$$

Here $\bar{f}_{0j}$ is proportional to the first nonvanishing derivative of $\bar{f}$ with respect to $\bar{y}_s$. If such derivative does not exist, then the corresponding term in (33) is zero. From (33) one finds, by integration over u;

$$\partial_t P(v,t) = o(1) \tag{34}$$

At this order of perturbation in ϵ, the v (or y) variable does not evolve in time. Hence, by combining (33) and (34) one obtains for the conditional probability $P(u\,|\,v;t)$:

$$\partial_t P(u\,|\,v;t) = \left\{ -\frac{\partial}{\partial u}\left[\lambda_1 u + \bar{f}_{0j}\epsilon^{j(1-b)-(1-a)}v^j\right] + \frac{\epsilon^{2a-1}}{2} Q_{xx}\frac{\partial^2}{\partial u^2}\right\} P(u\,|\,v;t)\left[1+o(1)\right] \tag{35}$$

This result describes how the fast variable u follows the slow variable v. If we are concerned with the evolution of the slow variable v only, it suffices to consider the stationary solution to which the distribution will relax on a time of the order of $O(1)$:

$$P_s(u\,|\,v) = \left(\frac{-\lambda_1\epsilon^{1-2a}}{\pi Q_{xx}}\right)^{1/2} \exp\left[\frac{\lambda_1\epsilon^{1-2a}}{Q_{xx}}\left(u + \frac{\bar{f}_{0j}}{\lambda_1}\epsilon^{j(1-b)-(1-a)}v^j\right)^2\right] \tag{36}$$

This allows to calculate the conditional averages (32) to dominant order in ϵ. Of course higher order correction to (36) have to be considered, if these conditional averages vanish. Note finally that it is also possible to calculate the conditional averages from the chain of the equations for the moments (see Ref. 30).

Further consideration of the result (36) and the specific values of a and b depend strongly on the nonlinearity of the model under consideration. Before proceeding with the general discussion, we illustrate some typical situations on explicit examples.

4. SOME EXAMPLES

In the following examples, we suppose for simplicity of notation that the (globally stable) macroscopic steady state is given by $\bar{x}_s = \bar{y}_s = 0$ with $\lambda_1 = -1$. We examine different cases, ranging from a strong coupling to a complete decoupling of the two variables.

4.1. Example A

Suppose we have

$$\partial_t x = -x + y^2 + \epsilon^{1/2}F_x \qquad \partial_t y = -xy^3 + \epsilon^{1/2}F_y \tag{37}$$

The general result (31) becomes in this particular case

$$\partial_t P(v;t) = \left\{ -\frac{\partial}{\partial v}\left[-\epsilon^{(1-a)+2(1-b)}\langle u\,|\,v\rangle v^3\right] + \frac{\epsilon^{2b-1}}{2} Q_{yy}\frac{\partial^2}{\partial v^2}\right\} P(v;t) \tag{38}$$

The conditional probability reads *to dominant order in* ϵ *for values* $\frac{1}{2} \leqslant a, b$

< 1 [cf. Eq. (24)]:

$$P_s(u\,|\,v) \sim \exp\left[-\frac{\epsilon^{1-2a}}{Q_{xx}} \left(u - \epsilon^{2(1-b)-(1-a)} v^2\right)^2 \right] \tag{39}$$

Let us try values of a and b such that

$$2(1-b) - (1-a) = 0 \tag{40}$$

One finds

$$\langle u\,|\,v\rangle = v^2 \tag{41}$$

In order to obtain a nontrivial result for $P(v; t)$, the noise and the systematic term in Eq. (38) have to be of the same order in ϵ (if not either the systematic term dominates on the noise or vice versa). This implies

$$(1-a) + 2(1-b) = 2b - 1 \tag{42}$$

which, together with (40), leads to

$$a = \tfrac{2}{3}, \qquad b = \tfrac{5}{6} \tag{43}$$

These values are consistant with the condition under which (39) was derived. Inserting the relations (43) in the expressions (38) and (39), one easily verifies that

$$\lim_{\epsilon \to 0} P_s(u\,|\,v) = \delta(u - v^2) \tag{44}$$

$$\lim_{\epsilon \to 0} P_s(v) \sim \exp\left(-\frac{v^6}{3Q_{yy}} \right) \tag{45}$$

$$\lim_{\epsilon \to 0} P_s(u) \sim \exp\left(-\frac{|u|^3}{3Q_{yy}} \right) \tag{46}$$

These results can be summarized as follows.

1. To dominant order in ϵ, the elimination of the fast variable amounts to setting $x = y^2$ in the equation for y. The value $b = 5/6$ then follows from the results for one variable systems. The critical variable thus exhibits amplified non-Gaussian fluctuations. It evolves on a slow time scale ($\tau = \epsilon^{2/3} t$).

2. A more surprising result is the non-Gaussian nature of the fluctuations of the fast variable. This can be explained as follows: on a time scale of $O(1)$, the fast variable relaxes to a Gaussian distribution, with the amplitude of its fluctuations $\langle \delta x^2 \rangle \sim O(\epsilon)$, centered around the average value y^2. These are the "intrinsic" fluctuations of x. On the slow time scale, the critical variable exhibits amplified fluctuations $\langle \delta y^2 \rangle \sim O(\epsilon^{1/3})$. The average of the "Gaussian" variable follows these slow fluctuations. This induces a supplementary contribution to the fluctuations of x of order:

$\langle \delta x^2 \rangle_{\text{ind}} \sim \langle \delta y^4 \rangle \sim O(\epsilon^{2/3})$. Hence the induced fluctuations are dominant over the intrinsic fluctuations of x.

The fact that the fluctuations of the fast variable reflect the critical properties of the slow variable may be of experimental interest if, for instance, the measurement of the critical variable proves to be difficult. The measurement of the "fast variable" can then yield the required information on the critical behavior of the system.

From the above discussion, we expect that for other nonlinearities of the system, the proper flucutations of the fast variable can be of the same order in ϵ as the fluctuations induced through the coupling to the slow variable, or even dominant on the latter. We proceed with the discussion of such cases.

4.2. Example B

Suppose we have

$$\partial_t x = -x + y^2 + \epsilon^{1/2} F_x \qquad \partial_t y = -xy + \epsilon^{1/2} F_y \tag{47}$$

Proceeding in the same way as for example A, we obtain

$$\partial_t P(v,t) = \left[-\frac{\partial}{\partial v} (\epsilon^{1-a} \langle u \,|\, v \rangle v) + \frac{\epsilon^{2b-1}}{2} Q_{yy} \frac{\partial^2}{\partial v^2} \right] P(v;t) \tag{48}$$

and

$$P_s(u \,|\, v) \sim \exp\left[-\frac{\epsilon^{1-2a}}{Q_{xx}} \left(u - \epsilon^{2(1-b)-(1-a)} v^2\right)^2 \right] \tag{49}$$

Again, we choose values of a and b such that

$$2(1-b) - (1-a) = 0 \tag{50}$$

Inserting the result $\langle u \,|\, v \rangle = v^2$ in (48) and identifying the ϵ dependence of the systematic and the noise term, we obtain the second equation:

$$1 - a = 2b - 1 \tag{51}$$

From (50) and (51) it follows that

$$a = \tfrac{1}{2}, \qquad b = \tfrac{3}{4} \tag{52}$$

For these values of a and b, the following results are obtained:

$$\lim_{\epsilon \to 0} P_s(u \,|\, v) \sim \exp\left[-\frac{1}{Q_{xx}} \left(u - v^2\right)^2 \right] \tag{53}$$

$$\lim_{\epsilon \to 0} P_s(v) \sim \exp\left(-\frac{v^4}{2Q_{yy}} \right) \tag{54}$$

$$\lim_{\epsilon \to 0} P_s(u) \sim \exp\left(-\frac{u^2}{2\langle \delta u^2 \rangle} \right) \tag{55}$$

with

$$\langle \delta u^2 \rangle = \frac{Q_{xx}(Q_{xx} + 4Q_{yy})}{2(Q_{xx} + 2Q_{yy})}$$

We conclude the following:

The adiabatic elimination again amounts to setting $x = y^2$ in the equation for y. The fluctuation of the fast variable are enhanced but their Gaussian nature is preserved, only the critical variable exhibits non-Gaussian fluctuations.

4.3. Example C

Suppose we have

$$\partial_t x = -x + y^4 + \epsilon^{1/2} F_x \qquad \partial_t y = -xy + \epsilon^{1/2} F_y \tag{56}$$

If we proceed in the same way as for examples A and B, we arrive at values $a = 1/3$, $b = 5/6$, which contradict the relation (24) ($a \geqslant 1/2$) from which the result (36) was derived. Therefore, the only possible nontrivial result for the conditional probability is obtained for

$$a = \tfrac{1}{2} \qquad \text{and} \qquad 4(1-b) - (1-a) > 0$$

hence

$$\lim_{\epsilon \to 0} P_s(u \mid v) \sim \exp\left(-\frac{u^2}{Q_{xx}} \right) \tag{57}$$

Since $\langle u \mid v \rangle = 0$ to this dominant order in ϵ, we have to calculate the correction to the result (57). We consider again the equation for $P(u, v; t)$ taking into account that $a = 1/2$:

$$\partial_t P(u, v; t) = \left[-\frac{\partial}{\partial u}\left(-u + \epsilon^{4(1-b)-1/2} v^4\right) - \frac{\partial}{\partial v}\left(\epsilon^{1/2} uv\right) \right.$$
$$\left. + \frac{1}{2}\left(\frac{\partial^2}{\partial u^2} Q_{xx} + \epsilon^{2b-1} \frac{\partial^2}{\partial v^2} Q_{yy} + 2\epsilon^{b-1/2} \frac{\partial^2}{\partial u \, \partial v} \right) \right] P(u, v; t) \tag{58}$$

One verifies that for a choice $b > 4/5$, this equation reduces, up to the first-order correction in ϵ, to

$$\partial_t P(u, v; t) = \left[-\frac{\partial}{\partial u}\left(u + \epsilon^{4(1-b)-1/2} v^4\right) \right.$$
$$\left. + \frac{1}{2} Q_{xx} \frac{\partial^2}{\partial u^2} \right] P(u, v; t)\left[1 + O(\epsilon)\right] \tag{59}$$

Hence, proceeding in the same way as for the derivation of (35),

$$\partial_t P(u|v;t) = \left[-\frac{\partial}{\partial u}(u + \epsilon^{4(1-b)-1/2}v^4) + \frac{1}{2} Q_{xx} \frac{\partial^2}{\partial u^2} \right] P(u|v;t)[1 + O(\epsilon)] \tag{60}$$

and the result (36) is found to remain valid up to the first-order correction in ϵ for $b > 4/5$. Substitution of the resulting value of $\langle u|v\rangle$ in the equation for $P(v;t)$ leads to

$$\partial_t P(v;t) = \left[-\frac{\partial}{\partial v}(\epsilon^{4(1-b)}v^5) + \frac{\epsilon^{2b-1}}{2} Q_{yy} \frac{\partial^2}{\partial v^2} \right] P(v;t) \tag{61}$$

Hence we find $b = 5/6$, which obviously satisfies the inequality $b > 4/5$. From the above results we can draw the following conclusion: To dominant order in ϵ, the fast variable is completely decoupled from the critical variable.

Note that in this example, the coupling of the slow variable to the fast variable is through the conditional average $\langle u|v\rangle$ and not through higher moments. For this reason, the flucutations of the fast variable do not affect the properties of the critical variable. Nevertheless, the adiabatic elimination again amounts to the substitution of $x = y^2$ in the equation y. It will be proven in the next section that this is indeed a general feature.

4.4. Example D

So far we have considered examples in which the equation for x always contained at least one term of the form y^k, $k \geqslant 1$. Let us now consider a case in which such a term does not exist:

$$\partial_t x = -x + x^k y^l + \epsilon^{1/2} F_x \qquad \partial_t y = -x^i y - y^3 + \epsilon^{1/2} F_y \tag{62}$$

with $k \geqslant 1$.

We know that the global stability of the macroscopic steady state implies the convergence of stochastic values to the deterministic values even for $t \to \infty$, i.e., $b < 1$. Therefore it is clear that

$$\frac{x^k y^l}{x} \sim o(1) \tag{63}$$

and is negligible for $\epsilon \to 0$. Hence, the above problem reduces, to dominant order in ϵ, to a problem of "external noise."[28] It is clear that the scaling value of x will be Gaussian: $a = 1/2$. This leads to the following equation for $P(v;t)$:

$$\partial_t P(v;t) = \left[-\frac{\partial}{\partial v}\left(-\epsilon\langle u^i \mid v\rangle v + \epsilon^{2(1-b)}v^3\right) + \frac{\epsilon^{2b-1}}{2} Q_{yy} \frac{\partial^2}{\partial v^2} \right] P(v;t) \tag{64}$$

So far, the situation is similar to that occurring in example C: to dominant order in ϵ, the x variable is decoupled from the y variable. We will now prove that the converse is also true, i.e., to dominant order in ϵ, the y variable is decoupled from the x variable. Since $b < 1$, we have $2b - 1 < 1$. Moreover, the conditional average $\langle u^i \mid v\rangle$ is of the order of $O(1)$ (or possibly vanishing). Therefore the systematic term $-\epsilon\langle u^i \mid v\rangle v$ is *always* negligible compared to the noise contribution. The equation for $P(v, t)$ then becomes

$$\partial_t P(v;t) = \left[-\frac{\partial}{\partial v}\left(\epsilon^{2(1-b)}v^3\right) + \frac{\epsilon^{2b-1}}{2} Q_{yy} \frac{\partial^2}{\partial v^2} \right] P(v;t) \tag{65}$$

and the value $b = 3/4$ follows. Note that the assumed global stability of the macroscopic steady state implies that the latter is unique. Therefore, the existence of a term of the form y^k, $k \geqslant 1$, in either the equation for x (examples A, B, C) or in the equation for y (present example D) is assured. In the last case, we showed that the latter term was dominant over all the mixed terms.

5. GENERAL DISCUSSION

In the previous examples, we found some common features such as the simple prescription for the elimination of the fast variable. The properties of the non critical variable were found to be qualitatively different depending on the specific nonlinearity of the model. We now present a general discussion which allows to classify the various possible cases according to the nonlinearity of the system.

The result (36) is valid to dominant order in ϵ for values of $1/2 \leqslant a, b < 1$. If we suppose that there exists a coefficient $\bar{f}_{0j} \neq 0$, three situations can be distinguished (the case $\bar{f}_{0j} = 0$ was amply discussed in example D, and will not be taken up again here):

1. $j(1-b) - (1-a) = 0$, with $a > 1/2$:

$$\lim_{\epsilon \to 0} P_s(u \mid v) = \delta\left(u + \frac{\bar{f}_{0j}}{\lambda_1} v^j\right) \tag{66}$$

hence

$$\langle u^k \mid v\rangle \sim \left(-\frac{\bar{f}_{0j}}{\lambda_1} v^j\right)^k$$

This case corresponds to the strong coupling of the two variables, as occurring in example A.

2. $j(1-b)-(1-a)=0$, with $a=1/2$:

$$\lim_{\epsilon \to 0} P_s(u \mid v) \sim \exp\left[\frac{\lambda_1}{Q_{xx}}\left(u+\frac{\bar{f}_{0j}}{\lambda_1} v^j\right)^2\right] \tag{67}$$

In this case the coupling of the x variable to the y variable is marginal (see example B). Note that the conditional average $\langle u \mid v\rangle$ is still independent of the force strength of the noncritical variable: $\langle u \mid v\rangle = (\bar{f}_{0j}/\lambda_1)v^j$.

3. $j(1-b)-(1-a)>0$, with $a=1/2$:

$$\lim_{\epsilon \to 0} P_s(u \mid v) \sim \exp\left(\frac{\lambda_1 u^2}{Q_{xx}}\right) \tag{68}$$

In this case, the x variable is decoupled from the y variable (see example C). Note that $\langle u^k \mid v\rangle = 0$ for k odd.

The case $j(1-b)-(1-a)<0$ need not to be considered since it leads to a non-normalizable conditional probability. Only in the case $j(1-b)-(1-a)=0$ do we always have nontrivial results for the conditional averages. Let us first consider the implications of such a choice. The conditional averages are then of the order of $O(1)$. The dominant systematic terms in the Eq. (31) for $P(v;t)$ are thus clearly those for which $k(l-a)+(l-1)(1-b)=(kj+l-1)(1-b)$ is minimal, i.e., the terms for which $kj+l$ is minimal. Having identified these dominant systematic terms, we note that the noise term should be of the same order in ϵ, in order to obtain a nontrivial result for $P(v;t)$; this implies

$$b=(kj+l)/(kj+l+1) \tag{69}$$

and consequently

$$a=(kj+l-j+1)/(kj+l+1) \tag{70}$$

These values of a and b are in accordance with $1/2 \leqslant a < b < 1$ when

$$k>1 \quad \text{or} \quad k=1 \quad \text{and} \quad l+1 \geqslant j \tag{71}$$

(note that $k+l \geqslant 2$ and $j \geqslant 2$). For $k>1$ or $k=1$ and $l+1>j$, one obtains values $1/2 < a, b < 1$, and the adiabatic elimination corresponds to setting $x=-\bar{f}_{0j}y^j/\lambda_1$, in the equation for y. For the limiting case $k=1$ and $l+1=j$ one finds $1/2=a<b<1$. Note that although in this case the intrinsic fluctuations of x affect the form of the conditional probability, they do not intervene in the adiabatic elimination since we need to calculate the conditional moment $\langle u \mid v \rangle$ only.

Let us now turn to the cases

$$k = 1 \qquad \text{and} \qquad l + 1 < j \tag{72}$$

In order to obtain a nontrivial result for $P(u \mid v; t)$ at the lowest order in ϵ, we take $a = 1/2$. To obtain a closed equation for the slow variable, we have to calculate the correction to the lowest-order result (68) since $\langle u \mid v \rangle$ is zero at this order of ϵ. As in example C, we can verify that for $a = 1/2$ and $b > j/j + 1$ the equation (33) for $P(u, v; t)$ remains valid to first-order correction in ϵ. Hence, the result (36) for the conditional probability remains valid to this order and one obtains

$$\langle u \mid v \rangle = -\frac{\bar{f}_{0j}}{\lambda_1} \epsilon^{j(1-b)-1/2} v^j \tag{73}$$

Using the above expression in the equation for the slow variable, one finds $b = (j + l)/(j + l + 1)$ which is in accordance with

$$b > j/(j + 1) \qquad (\text{indeed } l \geqslant 1)$$

As a conclusion, one can give the following recipe for the calculation of the asymptotic properties of the fluctuations in the vicinity of a cusp bifurcation:

1. Verify the global stability of the macroscopic steady state.

2. If the equation for the noncritical variable x does not contain a term depending only on y [i.e., $f(0, y) = 0$], then one can conclude that, to dominant order in ϵ, the two variables are decoupled. The equation for y should then contain a term depending only on y and the lowest-power term, say, y^j, is the dominant one. The appropriate scaling is then $a = 1/2$ and $b = j/(j + 1)$.

3. If the equation for x contains terms depending only on y (let y^j be the lowest power of these terms), then the adiabatic elimination amounts to setting

$$x = -\frac{\bar{f}_{0j} y^j}{\lambda_1} \tag{74}$$

in the equation of y. The dominant term in the latter equation will then be the term $x^k y^l$ for which $kj + l$ is minimal. The scaling exponent for the y variable is then

$$b = \frac{kj + l}{kj + l + 1} \tag{75}$$

4. The scaling exponent for x depends on the details of the nonlinearity:

$$a = \frac{kj + l - j + 1}{kj + l + 1} \qquad \begin{cases} \text{if } k > 1 \text{ or} \\ \text{if } k = 1 \text{ and } l + 1 > j \end{cases}$$

$$a = 1/2 \qquad \text{if } k = 1 \text{ and } l + 1 \leqslant j \tag{76}$$

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